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Complex Linear Groups of Degree at Most $v-3$*

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INTRODUCTION

The purpose of this paper is to prove the following theorems.

THEOREM 1. *Assume that G is a finite group, v is a prime greater than 7, and G has a faithful irreducible complex character Λ of degree at most $v-3$. Let V be a Sylow v subgroup of G. Then either $V \trianglelefteq G$ or $G/Z(G) \simeq PSL(2, v)$.*

Feit and Thompson [5] proved $V \trianglelefteq G$ if $\Lambda(1) < (v-1)/2$. Theorem 1 has been proved by Feit [7] if $|Z(G)|$ is odd and by the author [8] if $|Z(G)| \geqslant 12$ and $G = G'$.

THEOREM 2. *Let v be a prime larger than 7 and G be a finite group of order $v^a g'$, where $(v, g') = 1$. Assume that G has a faithful complex character Λ of degree at most $v-3$. Let V be a Sylow v subgroup of G. Then one of the following occurs:*

(A) $V \trianglelefteq G$

(B) *$G = HM$ where H and M are normal subgroups of G.*

(i) *$G/H \cong PSL(2, v)$,*

(ii) *H contains a normal subgroup V_0 of G, $|V_0| = v^{a-1}$ and $C_G(V_0) = V_0 \times M$.*

(iii) *If V_1 is a Sylow v subgroup of M, then $|V_1| = v$, $O_{v'}(C_M(V_1)) = M \cap H$, and $C_G(V_1) = V_1 \times H$.*

(iv) *If Λ is irreducible or $\Lambda(1) = (v \pm 1)/2$, then $H = Z(G)$.*

We note that Theorem 1 is certainly contained in Theorem 2, but for

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simplicity was stated separately. Theorem 2 was proved by Winter [16] if $A(1) < (2v+1)/3$.

The proof of Theorem 1 uses the classification of finite simple groups. Theoretically, the degrees of the irreducible characters of all the finite simple groups are known. However, this information does not seem to be readily available in a convenient form for direct use in the proof of Theorem 1 in some cases. Therefore, the proof of Theorem 1 does not in general quote character tables.

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If H is a finite group and q is a prime, let $|H_q|$ denote the order of a Sylow q-subgroup of H. In earlier papers about Theorem 1 the symbol p was used instead of v. The change from p to v was made because p is used in a different context in many of the references we will need for the proof of Theorem 1.

Throughout this section we assume that G satisfies the hypothesis of Theorem 1. Let $Z = Z(G)$, $|Z| = z$, $N = N_G(V)$, $C = C_G(V)$, $e = [N:C]$ and $t = (|v| - 1/e)$. Let $\bar{A}$ denote the image of a set A in $G/Z(G)$. For a positive integer k, $\phi(k)$ denotes the Euler function applied to k.

We generally use the notation of [10] except we use ${}^2D(q^2)$, ${}^2A(q^2)$, etc. for the twisted groups.

A group G satisfies *Hypothesis* 1 if G is a minimal counter example to Theorem 1.

LEMMA 1. *Assume G satisfies Hypothesis 1, then the following conditions hold*:

(i) $|V| = v$, $C = V \times Z$.

(ii) $A(1) = v - e$, $z \mid (v-e)$ *and* $2 \mid z$.

(iii) $G = G'$ *and* $\bar{G}$ *is simple.*

(iv) $\bar{N} = N_{\bar{G}}(\bar{V})$ *is a Frobenius group of order* ev. *Further*, e *is an odd integer*, $(v-1)/2 > e \geqslant 3$ *and* $e \mid \phi(v)$. *Moreover*, $v \geqslant 31$ *and* v *is the largest prime dividing* $|G|$.

Proof. The proof of [1, Proposition 2.1] shows that if G is a counterexample to Theorem 1, then there is another counterexample $G_1 \subseteq G$ such that A_{G_1} is irreducible, $|(G_1)_v| = v$, $G_1' = G_1$ and $G_1/Z(G_1)$ is simple. So minimality of $|G|$ implies that $G_1 = G$. Now [3] yields that $A(1) = v - e$, and $2 \mid z$ by [7, Theorem 1]. The rest of (i), (ii) and (iii), and the first statement in (iv), follow from [8, Lemma 1].

Now $e > 1$ by Burnside's transfer theorem, e is odd since $z \mid v - e$, and

$e < (v-1)/2$ by [16]. Also, $e \| |\mathrm{Aut}(V)| = \phi(v)$. By [1, Theorem 4] we have $v \geqslant \Lambda(1) + 3 \geqslant 31$.

Let h be the product of the orders of all Sylow q-subgroups of G where $q - 2 > \Lambda(1)$. Feit [6] implies that G has a normal abelian subgroup H of order h or h/q for some q dividing h. Since $z \mid \Lambda(1) < q$ (by Lemma 1(ii)), $(z, q) = 1$ and hence $(z, |H|) = 1$. If $h \neq v$, it follows that $H \simeq H/(H \cap Z) \simeq \bar{H}$ and $\bar{H}$ is a non-identity normal subgroup of $\bar{G}$ of odd order. However, $\bar{G}$ is simple of even order. Therefore, $h = v$ which implies that v is the largest prime dividing $|G|$.

LEMMA 2. *Assume that G satisfies Hypothesis* 1, *then* $v \nmid |X|$ *if X is any non-abelian simple group properly involved in* $\bar{G}$.

Proof. Assume that X is a non-abelian simple group properly involved in $\bar{G}$ such that $v \,\| |X|$. Then there are subgroups $Z \subseteq H \trianglelefteq J \subset G$ such that $J/H \simeq X$. Lemma 1(i) implies that we may choose notation so that $V \subseteq J$. Since Λ is a faithful character, Λ_J has an irreducible constituent Λ_1 such that $V \nsubseteq \ker \Lambda_1$. Let $K = \ker \Lambda_1$. Then $\Lambda_1(1) \leqslant \Lambda(1) \leqslant v - 3$ implies that the conclusion of Theorem 1 holds for J/K. So either $(J/K)/Z(J/K) \simeq PSL(2, v)$ or $VK/K \triangle J/K$. The former yields $|N_J(V) : C_J(V)| = |N_{J/K}(VK)/K) : C_{J/K}(VK/K)| = (v-1)/2$, whence $e \geqslant (v-1)/2$, which contradicts Lemma 1(iv). Since J/H is simple and $V \nsubseteq HK$, we have $K \subseteq H$. Thus if $VK/K \triangle J/K$ then $VH \triangle J$, whence $J = VH$ and $X \simeq V$, a contradiction.

Proof of Theorem 1. We assume that G satisfies Hypothesis 1. Thus $\bar{G} = G/Z(G)$ is simple and $G = G'$. Proposition 4.227 [10] implies that $Z(G)$ is a homomorphic image of the Schur multiplier of $\bar{G}$. Using Lemma 1(ii), we see that $\bar{G}$ is a simple group with a Schur multiplier of even order. The proof of Theorem 1 will proceed by a check of the finite simple groups. Let s be the order of a Sylow 2 subgroup of the Schur multiplier of $\bar{G}$, we will use Table 4.1 [10] and $s = 2^k \geqslant 2$ to determine possibilities for $\bar{G}$. Williams' [15] work on prime graph components will be used extensively to determine possible values for v.

(a) $\bar{G}$ is isomorphic to a group in Chev(p), p odd.

Assume (a) does not hold, then $\bar{G}$ is isomorphic to a sporadic group, $\bar{G} \in$ Chev(2), or $\bar{G} \simeq \mathrm{Alt}(n)$, $n \geqslant 5$. Now $s = 2^k \geqslant 2$, v is the largest prime dividing $|G|$ and $v \geqslant 31$ by Lemma 1. Table 4.1 [pp. 302–303, 10] and Table 2.4 [pp. 135–136, 10] may now be used to see that $\bar{G} \simeq F_2$ and $v = 47$ or $\bar{G} \simeq \mathrm{Alt}(n)$. If $\bar{G} \simeq F_2$, then Lemma 1(iv) implies $e \geqslant 3$ and e is an odd divisor of $\phi(47) = 2 \cdot 23$. Therefore, $e = 23$, but this contradicts $e < (v-1)/2$. If $\bar{G} \simeq \mathrm{Alt}(n)$, $n \geqslant 5$, then $v \geqslant 31$ implies $n \geqslant 31$. Since $\mathrm{Alt}(n-1)$ is properly involved in $\mathrm{Alt}(n)$, Lemma 2 implies $v \nmid (n-1)!$. Thus, $n = v$ and $\bar{G} \simeq \mathrm{Alt}(v) \subseteq \mathrm{Sym}(v)$. Viewing $\bar{G}$ as a subgroup of $\mathrm{Sym}(v)$, we see that $[N_{\mathrm{Sym}(v)}(\bar{v}): C_{\mathrm{Sym}(v)}(\bar{V})] = [N_{\mathrm{Sym}(v)}(\bar{V}):\bar{V}] = v-1$. It follows that $e = [N_{\mathrm{Alt}(v)}(\bar{V}):\bar{V}] \geqslant (v-1)/2$. This contradicts Lemma 1(iv).

(b) $\bar{G} \not\simeq A_l(q)$, $l \geqslant 2$, q odd.

Assume $\bar{G} \simeq A_l(q)$, $l \geqslant 2$ and q is odd. Williams' [*pp.* 489–492, 15] implies that either $l+1$ is a prime and $v = (q^{l+1}-1)/(q-1)(l+1, q-1)$ or l is a prime and $v = (q^l-1)/(q-1)$. By Table 4.1[10], $s \mid (l+1, q-1)$. Since $s = 2^k \geqslant 2$ and $l \geqslant 2$, we may assume that l is an odd prime and $v = (q^l-1)/(q-1)$. However, $A_{l-1}(q)$ is involved as a Levi factor in $A_l(q)$, and $v \mid |A_{l-1}(q)|$. We recall that $A_{l-1}(q) \simeq PSL(l,q)$ is simple for $l \geqslant 3$. This contradicts Lemma 2.

(c) $\bar{G} \not\simeq B_l(q)$ or $C_l(q)$, $l \geqslant 2$, q odd.

Assume $\bar{G} \simeq B_l(q)$ or $C_l(q)$ for $l \geqslant 2$ and q odd. Lemma 1 implies $|\bar{V}| = v$ and $\bar{N}$ is a Frobenius group of order ev. Therefore, $(v, q) = 1$ and $\bar{V}$ is a maximal torus of $\bar{G}$. Temporarily using the notation of [4], we may view $\bar{V}$ as the torus T_w associated with some element w of the Weyl group W of $\bar{G}$. For $\bar{G} \simeq B_l(q)$ or $C_l(q)$, W may be described in the following manner [11, p. 64]. W acts as the group of all permutations and sign changes of an orthonormal basis of R^l, and so contains -1. As noted on p. 46[4], $e = [N_{\bar{G}}(T_w): T_w]$ is divisible by $|C_W(w)|$. Since $|Z(w)|$ divides $|C_W(w)|$, this contradicts e odd.

(d) $\bar{G} \not\simeq D_l(q)$, $l \geqslant 4$, q odd.

If $\bar{G} \cong D_l(q)$, $l \geqslant 4$, and q odd, then Williams' [15] implies one of the following set of conditions is satisfied:

(i) $\bar{G} \cong D_l(3)$, l is a prime, $v = (3^l-1)/2$;
(ii) $\bar{G} \cong D_l(3)$, $l-1$ is a prime, $v = (3^{l-1}-1)/2$, or
(iii) $\bar{G} \cong D_l(5)$, l is a prime, and $v = (5^l-1)/4$.

However, $A_{l-1}(q)$ is involved in $\bar{G}$ as a Levi factor and $(q^{l-1}-1)(q^l-1) \| |A_{l-1}(q)|$. But $A_{l-1}(q)$ is simple since $l \geqslant 4$. This contradicts Lemma 2.

(e) $\bar{G} \not\simeq {}^2A_l(q^2)$, $l \geqslant 2$, q odd.

Assume that $\bar{G} = {}^2A_l(q^2) \simeq PSU(l+1, q)$ where q is odd. Since s is even, Table 4.1 [10] implies that l is odd. Therefore, we may use [15] to see that l is a prime and $v = (q^l+1)/(q+1)$. However, ${}^2A_{l-1}(q^2)$ is involved in ${}^2A_l(q^2)$ [p. 138, 14] and $(q^l+1)/(q+1) \| |{}^2A_{l-1}(q^2)|$. ${}^2A_{l-1}(q^2)$ is simple when $l \geqslant 3$ and q is odd. This contradicts Lemma 2.

(f) $\bar{G} \not\simeq {}^2D_l(q^2)$, $l \geqslant 4$, and q odd.

Assume that $G \cong {}^2D_l(q^2)$, $l \geqslant 4$, and q odd. Since s is even [15] implies that one of the following set of conditions is satisfied:

(i) $\bar{G} \cong {}^2D_l(3^2)$, $l = 2^n + 1$, l is not a prime, $v = (3^{l-1} + 1)/2$;

(ii) $G \cong {}^2D_l(3^2)$, l is a prime, $l \neq 2^n + 1$, $v = (3^l + 1)/4$;

(iii) $\bar{G} \cong {}^2D_l(3^2)$, $l = 2^n + 1$, l is a prime, $v = (3^{l-1} + 1)/2$ or $v = (3^l + 1)/4$; or

(iv) $\bar{G} \cong {}^2D_l(q^2)$, $l = 2^n$, $v = (q^l + 1)/2$.

If $\bar{G} \cong {}^2D_l(3^2)$, where l is odd, then $B_{l-1}(3)$ and ${}^2A_{l-1}(3^2)$ are involved in $\bar{G}$ [p. 138, 14]. However, $(3^{l-1} + 1)/2 \,||\, |B_{l-1}(3)|$ and $(3^l + 1)/4 \,||\, |{}^2A_{l-1}(3^2)|$ when l is odd. This contradicts Lemma 2 since $l \geqslant 5$. Thus, we may assume that $l = 2^n$, $\bar{G} \simeq {}^2D_l(q^2)$ and $v = (q^l + 1)/2$.

Recall that ${}^2D_l(q^2) \simeq P\Omega^-(2l, q)$. Hence, there is a vector space M of dimension $2l$ over $GF(q)$, the field of q-elements, such that $\bar{N}$ acts faithfully on M. Suppose that M_1 is a non-trivial proper subspace of M which is stabilized by $\bar{V}$. Since $\bar{N}$ acts faithfully on M, $\bar{V}$ acts non-trivially on at least one of M_1 and M/M_1. It follows that $v \,||\, |GL(k, q)|$ or $v \,||\, |GL(2l - k, q)|$ where $\dim_{GF(q)} M_1 = k$. Thus, $v \mid q^t - 1$ for some $t \leqslant \max(k, 2l - k)$. Lemma 7 [15] and $v = (q^l + 1)/2$ with $l = 2^n$ provide a contradiction. Therefore, $\bar{V}$ acts irreducibly on M. Proposition 19.8 [13] implies that $\bar{N}$ may be viewed as a subgroup of $T(q^{2l})$. $T(q^{2l})$ is defined on p. 229 [13]). Further, $C_{\bar{N}}(\bar{v}) = \bar{V}$ is contained in the subgroup $T_1(q^{2l})$ of $T(q^{2l})$ consisting of linear transformations, i.e., $\sigma = 1$. It is easy to see that $T_1(q^{2l})$ is abelian. Therefore, $e = [\bar{N} : \bar{V}] \,|\, [T(q^{2l}) : T_1(q^{2l})] = 2l$ (see p. 229 [13]). However, $l = 2^n$ now implies $e \mid 2^{n+1}$. This contradicts Lemma 1(iv).

(g) $\bar{G} \not\simeq E_7(q)$, q odd.

Assume $\bar{G} \simeq E_7(q)$, then [15] implies that $q = 3$ and $v = 757$ or 1093. However, $E_7(3)$ involves the groups $E_6(3)$ and $A_6(3)$ as Levi factors. Now $1093 = (3^7 - 1)/2$ divides $|A_6(3)|$ and $757 = 3^6 + 3^3 + 1$ divides $|E_6(3)|$. This contradicts Lemma 2.

(h) $\bar{G} \neq E_8(q)$, $E_6(q)$, $F_4(q)$, $G_2(q)$, ${}^2B_2(q)$, ${}^2E_6(q^2)$, ${}^2F_4(q)$, ${}^2G_2(q)$ or ${}^3D_4(q)$, q odd.

Table 4.1 [10] implies the Schur multiplier is odd for all the groups listed in (h).

Parts (a)–(h) and the classification of finite simple groups imply that the only possibility for $\bar{G}$ is $\bar{G} \cong A_1(q)$ and $q \neq v$, q odd. The character tables for $PSL(2, q)$, q odd appear in [12]. If $q = p^k$, a check of these tables shows that every odd p'-element is real. This contradicts Lemma 1(iv). Therefore, Hypothesis 1 is empty so Theorem 1 is proved.

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Throughout this section we assume that the hypothesis of Theorem 2 is satisfied. Since $\Lambda(1) \leqslant v - 3$, Λ_V is a sum of linear characters of V. Thus,

$V' \subseteq \ker \Lambda$ and Λ faithful imply that V is abelian. The following results are found in [16] and will be useful in the remainder of Section 2.

(2.1) Let T be a finite group which has a faithful complex character Λ of degree at most $v-2$, v a prime. If T has a normal v-complement, then $T = T_v \times O_{v'}(T)$.

(2.2) Let v be a prime and T a finite group which has a faithful complex character Λ all of whose irreducible constituents have degree less than $(v-1)/2$. Then T has an abelian normal Sylow v-subgroup.

Feit [6] has shown that G has a normal subgroup V_0 where $|V_0| = v^{a-1}$. We will assume that $V \not\trianglelefteq G$ and show Theorem 2(B) is satisfied.

It follows from (2.2) that we may assume $(v-1)/2 \leqslant \Lambda(1) \leqslant v-3$, and $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_1 is an irreducible constituent with $\Lambda_1(1) \geqslant (v-1)/2$, and every irreducible constituent of Λ_2 (if indeed $\Lambda_2 \neq 0$) has degree less than $(v-1)/2$. Let $K = \ker \Lambda_1$. Then Λ_1 is a faithful irreducible complex character of G/K and $\Lambda_1(1) \leqslant v-3$. We wish to show that $G/K/Z(G/K) \simeq PSL(2, v)$. If not, then Theorem 1 implies that $VK \trianglelefteq G$. However, $K \cap \ker \Lambda_2 = \{1\}$ since Λ is faithful. Applying (2.2) to $\Lambda_{2|_K}$ yields $K_v \trianglelefteq K$. Thus, $K_v \trianglelefteq VK$ whence $C_{KV}(K_v) \trianglelefteq VK$. Since $C_K(K_v) = K_v \times O_{v'}(C_K(V))$ and VK/K is a v-group, $C_{VK}(K_v)$ has a normal v-complement. Hence, we may apply (2.1) to $C_{VK}(K_v)$ to see that $C_{KV}(K_v) = V \times O_{v'}(C_{KV}(K_v))$. Now $V \trianglelefteq VK$ follows from $C_{VK}(K_v) \trianglelefteq VK$. Again noting that V is a characteristic subgroup of VK and $VK \trianglelefteq G$, we see that $V \trianglelefteq G$. This is a contradiction. Therefore, $G/K/Z(G/K) \simeq PSL(2, v)$.

Let H be the normal subgroup of G such that $H \supseteq K$ and $H/K = Z(G/K)$. It follows from the previous paragraph that $G/H \simeq G/K/Z(G/K) \simeq PSL(2, v)$. In particular, H is a maximal normal subgroup of G. Recall that V_0 is a normal subgroup of G of order v^{a-1}. Now $G/H \simeq PSL(2, v)$ implies that $HV_0 \neq G$. Since $HV_0 \trianglelefteq G$, the maximality of H yields $V_0 \subseteq H$. Now $|PSL(2, v)_v| = v$ implies $H_v = V_0$.

Let $G_0 = C_G(V_0)$, then $G_0 \trianglelefteq G$ and $V \subseteq G_0$. If $V \subseteq Z(N_{G_0}(V))$, then Theorem 7.4.4 [9] implies that G_0 has a normal v-complement. Applying (2.1) to Λ_{G_0} and G_0 yields $G_0 = V \times O_{v'}(G_0)$. It follows that $V \trianglelefteq G$, a contradiction. Since $|V_0| = |V|/v$, Theorem 7.4.4 [9] now implies $V_0 = Z(N_{G_0}(V)) \cap V$, $V = V_0 \times V_1$ where $V_1 = G_0' \cap V = (N_{G_0}(V))' \cap V$, and the maximal v-factor group of G_0 is isomorphic to V_0. Let M be the normal subgroup of G_0 such that $M \supseteq G_0'$ and $G_0/M \simeq V_0$, then $G_0 = M \times V_0$. We note that $M = O^v(G_0)$ so that M is a normal subgroup of G. Further, $|M_v| = v$ and $M_v \not\trianglelefteq M$ since $V \not\trianglelefteq G$.

MH is a normal subgroup of G with Sylow v subgroup V. Thus, $MH \neq H$ and $G = MH$ follows from the maximality of H. We note that B(i) and (ii) have been proved.

Let V_1 be an arbitrary Sylow v subgroup of M, then $|V_1| = v$. Therefore,

$M/(M \cap H) \simeq G/H \simeq PSL(2, v)$ implies that $M \cap H$ is a normal v'subgroup of G. Applying (2.1) to $V_1(M \cap H)$ and $\Lambda_{V_1(M \cap H)}$yields $M \cap H \subseteq O_{v'}(C_M(V_1))$. Now $M \cap H = O_{v'}(C_M(V_1))$ follows from the structure of the normalizer of a Sylow v subgroup in $PSL(2, v)$. Let $V_1 = \langle x \rangle$ and $h \in H$, then $x^{-1}x^h \in [M, H] \subseteq M \cap H = O_{v'}(C_M(V_1))$. Therefore, $x^h = xy$, where $y \in O_{v'}(C_M(V_1))$. It easily follows that $y = 1$. Therefore, $H \subseteq C_G(V_1)$. Now $C_M(V_1) = V_1 \times M \cap H$ and $G = HM$ imply $C_G(V_1) = V_1 \times H$. Thus, (iii) is proved.

If Λ is irreducible (which must be the case if $\Lambda(1) = (v-1)/2$), then Theorem 1 implies that $G/Z(G) \simeq PSL(2, v) \simeq G/H$. Now $H/K = Z(G/K)$ implies that $Z(G) \subseteq H$. Thus $Z(G) = H$.

If Λ is reducible and $\Lambda(1) = (v+1)/2$, then $\Lambda_1(1) = (v-1)/2$ and $\Lambda_2(1) = 1$. Let $L = \ker \Lambda_2$, so that G/L is abelian. Then $K \cap L = \langle 1 \rangle$ implies that G imbeds in $G/K \times G/L$ via $g \to (gK, gL)$, and the image of H is central. It follows that $H = Z(G)$. Now (iv) is proved and the proof of Theorem 2 is complete.

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